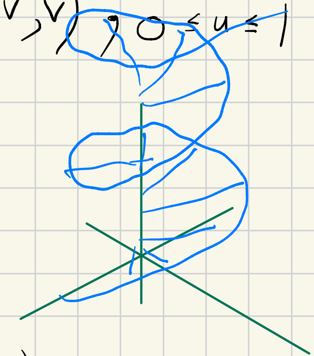
Pre-class Warm-up!!!

1. What surface does the function

$\Phi(u,v) = (u\cos v, u\sin v, v)$, $0 \le u \le 1$, $0 \le v \le 2\pi$

parametrize?



a. a cone

b. a cylinder

c. a helicoid (a spiral ramp) ✓

d. a sphere

e. None of the above.

2. Which integral computes its area?

a. $\int_0^{2\pi}\int_0^1 \|\Phi\| \, du \, dv$

b. $\int_0^{2\pi}\int_0^1 (T_u \times T_v) \cdot dS$

c. $\int_0^{2\pi}\int_0^1 \|T_u \times T_v\| \, du \, dv$ ✓

d. $\int_0^{2\pi}\int_0^1 du \, dv$

e. None of the above

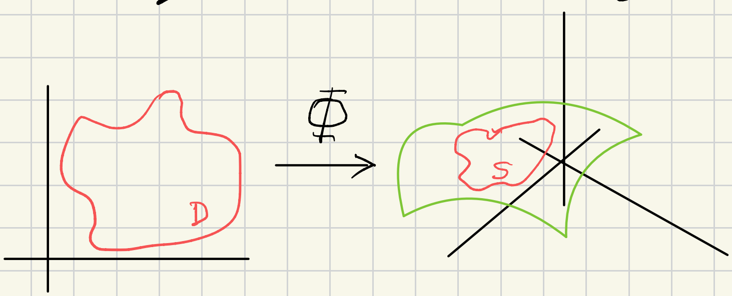
Section 7.5 Surface integrals of functions.

We learn:

- the same as in section 7.4 (which was about finding surface area) made a bit more complicated by integrating a scalar-valued function over some surface.
- We don't really need page 396: Integrals over graphs.

Recall: the area of a surface S parametrized by a function (x,y,z) =Phi(u,v) is given by

$$A = \iint_D \|T_u \times T_v\| \, du\,dv = \iint_S dS$$



$$T_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \quad T_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

New formula. We suppose we have a function f : R^3 -> R .

$$\iint_S f \, dS = \iint_D f(\Phi(u,v)) \, \|T_u \times T_v\| \, du\,dv$$

new notation

Interpretation:

1. We extend integrals from integrating over a flat region of $\mathbb{R}^2$ to a curvy region.

2. If the surface is a film of varying density $f(x,y,z)$ at (x,y,z) in gm/cm^2 then the mass of the film is

$$\iint_S f \, dS$$

Like question 6:

Evaluate $\displaystyle\iint_S xz\, dS$

where S is the part of the plane z = 1 + 2x - 3y that lies inside the cylinder x^2 + y^2 = 4.

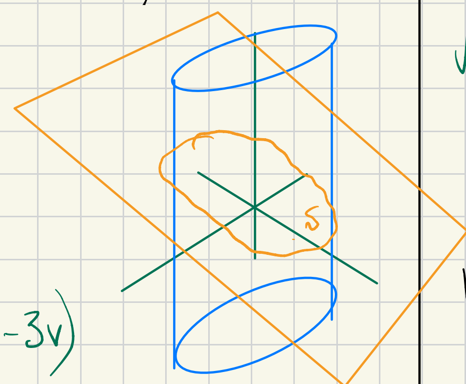
Solution.

Step 1: Parametrize S, which is the graph of $f(x,y) = 1+2x-3y$

Take

$$\Phi(u,v) = (u, v, 1+2u-3v)$$

$$0 \le u^2+v^2 \le 4$$



Step 2 $T_u = (1,0,2)$ $T_v = (0,1,-3)$

$T_u \times T_v = (-2, 3, 1)$, $\|T_u \times T_v\| = \sqrt{4+9+1} = \sqrt{14}$

Step 3 The integral is

$$\iint_{0 \le u^2+v^2 \le 4} u(1+2u-3v)\sqrt{14}\, du\, dv$$

Step 4 Use polar coordinates (r,θ)

$$\int_0^{2\pi}\int_0^2 r\cos\theta(1+2r\cos\theta-3r\sin\theta)\sqrt{14}\, r\, dr\, d\theta$$

$= \ldots$

What about $\displaystyle\iint_S x\, dS$?

Example: A unit sphere is made of material with surface density z^2 gm/cm^2 at point (x,y,z). Find the mass of the sphere.

Solution: Evaluate $\iint_{\text{sphere}} z^2 \, dS$

Phi(u,v) = (sin v cos u, sin v sin u, cos v)
0 ≤ u ≤ 2π, 0 ≤ v ≤ π

T_u = (- sin v sin u, sin v cos u, 0)
T_v = (cos v cos u, cos v sin u, -sin v)

same as before

T_u x T_v = (-sin^2 v cos u, -sin^2 v sin u, -sin v cos v)

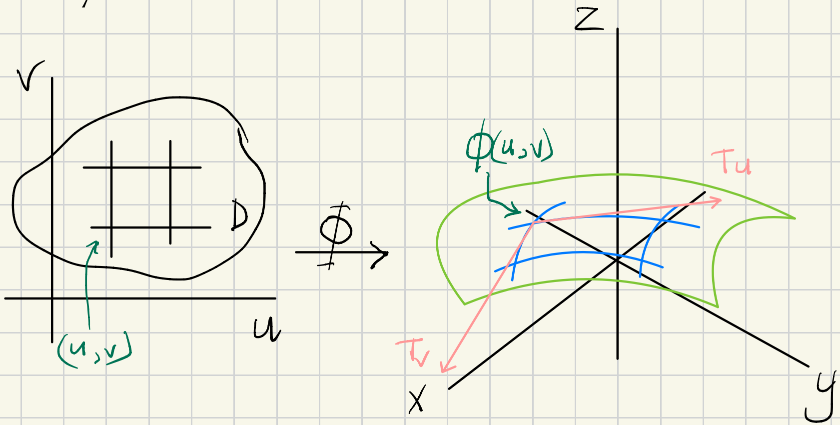
|| T_u x T_v || = $\sqrt{\text{sin^4 v + sin^2 v cos^2 v}}$ = sin v

Mass = $\int_0^{2\pi}\int_0^{\pi} \cos^2 v \cdot \sin v \, dv \, du = \ldots$

Question: what is z when expressed in (u,v)-coordinates?

a. sin v
✓ b. cos v
c. tan v
d. sin u
e. cos u

Why it works:



The scaling factor by which the parallelogram area in the (x,y,z) space is multiplied from the the little square in the (u,v,) plane, is

|| T_u x T_v ||